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Nash Bosonization in Path Integral for Quantum Riemannian Geometry

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Abstract We propose an effective quantum theory for Einstein Gravitation theory by making use of the Nash Theorem of Riemann Metrics parametrized by immersions. We show the usefulness of this phenomenological (low energy) path integral bosonization scheme through the evaluation for the Classical Newton potential by means of a Wilson loop path integral evaluation.

Keywords Quantum gravity · Path integral bosonization · Wilson loops in quantum gravity

1 Introduction

One of the most challenge mathematical problems in modern field theory is certainly the problem of choice of the correct dynamical variable to be quantized (or path integrated) in the theory of Random Geometry of metric fields in a given (fixed) manifold M. Several frameworks on the last decades have been proposed, however without producing yet a consistent quantum field theoretic framework, useful to implement evaluations outside the usual (non-renormalizable) coupling constant perturbation Feynmann-Dyson scheme.

In this Brief Report we intend to contribute for such a difficult problem of quantizing Quantum Gravity by proposing as suitable variables to be quantized on phenomenological grounds, the field of the immersions applications of a given manifold of dimension n in a convenient ambient extrinsic Euclidean Space R^d (with $d > n$): The famous Whitney and Nash embeddings/immersion-embeddings theorems applied to our C^∞ space-time manifold M where the dynamics takes place. These ideas are proposed in this note and can be considered as an approximate Bosonization of the usual metric variable theory in terms of "stress-strain" degrees of freedom associated to the Nash parametrization of the metric tensor (a quantum elasticity field theory for Einstein Gravitation theory).

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We show the usefulness of this phenomenological path integral scheme for Quantum Riemannian Geometry, by evaluating straightforwardly the Classical Newton Potential by means of a Wilson Loop evaluation associated to a static trajectory of a pair of massive particle and quantum averaged in an effective induced quantum gravity dynamics of fermionic matter at the leading semi-classical limit of $c \to \infty$ (here c denotes the light-velocity parameter).

2 Quantum Riemannian Geometry as a Dynamics of Bosonic Quantum Immersions and the Newton Gravitation Law

Let us start this section by recalling the Nash Theorem that asserts that every Riemannian metric in a C^∞-manifold M $\{g_{\mu\nu}(x)\}$ (a $C^2(M)$-tensor field) can be always obtained from an immersion $f^A\colon M \to R^{s(d)}$ ($f^A \in C^1(M)$ and rank $D_x f = d$) in a suitable Euclidean space $R^{s(d)}$, here the dimension of the Euclidean ambient space is strictly greater than d (a better lower bound is given by the inequality $s(D) \geq 2d - 1$) ([1])

$$g_{\mu\nu}(x) = \sum_{A=1}^{s(d)} \frac{\partial f_A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu} = \frac{\partial f^A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu}. \tag{1}$$

We would thus expect that in this vectorial like bosonization all equations and path-integrals in Riemannian Geometry should acquires a more invariant and suitabe expressions for analysis. Let us thus set up some formulae related to this new metrical variable parametrization as pointed out by (1).

Let us consider the context of an effective scheme, where one should consider "lenght" scales appropriated for the governing quantum dynamics under analysis. In this context it appears important to consider already built in the formulae, the important non-relativistic limit represented by the hypothesis of the analycity of the geometrical objects in relation to the inverse of light velocity. As a consequence one should envisages an expansion in powers of $\frac{1}{c}$ for the Nash scalar immersion fields

$$f_A(x^\gamma) = x^\alpha \delta_A^\alpha + \sum_{\ell=1}^{\infty}\left(\frac{1}{c}\right)^\ell \varphi_A^\ell(x^\gamma). \tag{2}$$

The metrical variable takes the simple form at the leading $c \to \infty$ limit:

$$\begin{aligned} g_{\mu\nu}(x^\gamma) &= \left(\delta_\mu^A + \frac{1}{c}\frac{\partial}{\partial x^\mu}\varphi_A^{(1)}\right)\left(\delta_A^\nu + \frac{1}{c}\frac{\partial}{\partial x^\nu}\varphi_A^{(1)}\right) \\ &= \delta_{\mu\nu} + \frac{1}{c}\left(\frac{\partial}{\partial x^\mu}\varphi_\nu^{(1)} + \frac{\partial}{\partial x^\nu}\varphi_\mu^{(1)}\right)(x^\gamma), \end{aligned} \tag{2-a}$$

$$g^{\mu\nu}(x^\gamma) = \delta_{\mu\nu} - \frac{1}{c}\left(\frac{\partial \varphi_\nu^{(1)}}{\partial x^\mu} + \frac{\partial}{\partial x^\nu}\varphi_\mu^{(1)}\right)(x^\gamma). \tag{2-b}$$

The Christofell connections are straightforwardly computed at this leading limit and take the very simple form

$$\Gamma_{\alpha\beta}^\mu(x) = \frac{1}{2} g^{\mu\gamma}\left(\frac{\partial}{\partial x^\alpha} g_{\beta\gamma} + \frac{\partial}{\partial x^\beta} g_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma} g_{\alpha\beta}\right)$$

$$= \frac{1}{c}\frac{\partial^2 \varphi_\mu^{(1)}(x^\gamma)}{\partial x^\alpha \partial x^\beta} + O\left(\frac{1}{c^2}\right). \tag{3}$$

The Riemann four-tensor is simply given by

$$R^\mu_{\gamma,\alpha\beta}(x) = \frac{1}{c^2}\left\{\frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\alpha \partial x^{\gamma'}}\frac{\partial^2 \varphi_{\gamma'}^{(1)}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\beta \partial x^{\gamma'}}\frac{\partial^2 \varphi_{\gamma'}^{(1)}}{\partial x^\gamma \partial x^\alpha}\right\} + O\left(\frac{1}{c^4}\right), \tag{4-a}$$

which produces the following expression for the Ricci tensor

$$R_{\alpha\beta}(x) = R^\mu_{\alpha,\mu\beta} = \frac{1}{c^2}\left\{\frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\mu \partial x^{\gamma'}}\frac{\partial^2 \varphi_{\gamma'}^{(1)}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\beta \partial x^{\gamma'}}\frac{\partial^2 \varphi_{\gamma'}^{(1)}}{\partial x^\alpha \partial x^\mu}\right\} + O\left(\frac{1}{c^4}\right), \tag{4-b}$$

and the associated scalar of curvature

$$R(x) = \left(g^{\beta\alpha}R_{\alpha\beta}\right)(\xi) = \frac{1}{c^2}\left[\frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\mu \partial x^{\gamma}}\frac{\partial^2 \varphi_{\gamma}^{(1)}}{\partial x^\beta \partial x^\beta} - \frac{\partial^2 \varphi_\mu^{(1)}}{\partial x^\beta \partial x^{\gamma}}\frac{\partial^2 \varphi_{\gamma}^{(1)}}{\partial x^\beta \partial x^\mu}\right]. \tag{4-c}$$

At the quantum geometrical level the functional-path integral measure leads to the usual Feynman path integral measure as defined by the $c \to \infty$ leading Nash immersion fields $\{\varphi_\mu^{(1)}\}_{\mu=1,\dots,d}$ as one can see from the simple variable change written below

$$\begin{aligned} ds^2 &= \int_M d^D x\left\{\sqrt{g}\,\delta g_{ab}\left(g^{aa'}g^{bb'} + g^{ab}g^{a'b'}\right)\delta g_{a'b'}\right\}(x) \\ &= \frac{1}{c^2}\int_M d^D x\left[(\delta\varphi_\mu^{(1)})\left(-\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right)(\delta\varphi_\nu^{(1)}\right](x) \end{aligned} \tag{5}$$

and thus

$$\begin{aligned} d\mu[g_{\alpha\beta}] \cong D^F[\varphi_\mu^{(1)}] = \Bigg\{\prod_{\mu=1}^{D}\left(\prod_{x\in M} d\varphi_\mu^{(1)}(x)\right) \\ \times \det{}^{-\frac{1}{2}}\left[-\frac{2}{c^2}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\right]. \end{aligned} \tag{6}$$

At this point is worthing call the reader attention that next $\frac{1}{c}$-corrections can be easily taken into account in the formulae above written generating now a fixed degree polinomial non-linearity on then and a non-trivial Faddev-Popov determinant in the new product Feynman measure equation (6).

We take as the weight for our Wilson Loop averages in our leading Nash fields a higher order Einstein-Hilbert action as given by an the effective action obtained after integrating out a massive femionic matter field at the limit of large mass [2]

$$\begin{aligned} &\det\left[\not{\partial} + \Gamma^\mu_{\alpha\beta}\sigma^{\alpha\beta} + m\right]_{m\to\infty}^{c\to\infty} \\ &\quad \sim \lim_{c\to\infty}\left\{\exp\left\{-\frac{1}{2G^{\text{eff}}}\int_M d^\nu x\left[\sqrt{g}\,\Gamma^\mu_{\alpha\beta}g^{\alpha\alpha'}(-\Delta)g^{\beta\beta'}\Gamma^\mu_{\alpha'\beta'}\right](x)\right\}\right\} \\ &\quad = \exp\left\{-\frac{1}{8\pi G^N}\int_M d^\nu x\left[\varphi_\alpha^{(1)}(-\Delta)^3\varphi_\alpha^{(1)}\right]\right\}. \end{aligned} \tag{7}$$

Here G^N is the (somewhat effective) Newton Gravitation constant.

Let us deduce the Newton Gravitation Law from the above written formulae in terms of the Nash field.

In the Riemannian quantum geometry, the above written Holonomy factor defined by the $SO(d)$-valued vector field $\Gamma^{\mu}_{\alpha\beta}(x)\sigma^{\alpha\beta}$, here $\sigma^{\alpha\beta}$ are the generators of the $SO(D)$ Group (the Euclidean Lorentz Group) is expected to lead to the Newton law in the non-relativistic and dimension mean-field limits $D \to \infty$ evaluation of its quantum average for a static (non-fluctuating) trajectory

$$\begin{aligned}\langle W[C_{(R,T)}]\rangle &\sim \frac{1}{Z}\int D^F[\varphi^{(1)}_{\mu}(x)]\exp\left\{-\frac{1}{8\pi G_N}\int d^D x\left[\varphi^{(1)}_{\alpha}(-\Delta)^3\varphi^{(1)}_{\alpha}\right]\right\}(x)\\ &\quad\times\frac{1}{D}\mathrm{Tr}_{SO(D)}\left\{\mathbb{P}\left[\exp i\left(\oint_{C_{(R,T)}}\Gamma^{\mu}(C(\sigma))\dot{C}_{\mu}(\sigma)d\sigma\right)\right]\right\}\end{aligned} \tag{8}$$

here $\mathbb{P}$ is path $SO(D)$-indexes ordination operator along the static trajectory $C_{(R,T)} = \{C_{\mu}(\sigma), 0 \le \sigma \le T\}$ and given by the boundary of a rectange $\{-\frac{T}{2} \le x^0 \le \frac{T}{2}, -\frac{R}{2} \le x^1 \le \frac{R}{2}\}$.

The Newton gravitation potential should be given by the lowest quantum energy state associated to the quantum propagation of the gravitation interacting pair and it is given explicitly by the ergodic-temporal (non-relativistic) limit of (8) ([3])

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\ell g\{\langle W[C_{(R,T)}]\rangle\}. \tag{9}$$

In order to evaluate the quantum non-abelian Holonomy factor (8) at the Gravitation mean field limit $D \to \infty$, as much as similar calculations done in Yang-Mills Theory ([3]), we write the Holonomy path ordered object as the one-dimensional fermion (Grasmanian variables) living on the contour $C_{(R,T)}$

$$\begin{aligned}&\frac{1}{D}\mathrm{Tr}_{SO(D)}\left\{\mathbb{P}\left[\exp i\oint_{C_{(R,T)}}\Gamma^{\mu}_{\alpha\beta}(C(\sigma))(\sigma^{\alpha\beta})\dot{C}_{\mu}(\sigma)\right]\right\}\\ &=\int_{\substack{\theta_{\alpha}(0)=\theta_{\alpha}(T)\\ \theta^*_{\alpha}(0)=\theta^*_{\alpha}(T)}}\prod_{\sigma\in[0,T]}(d\theta_{\alpha}(\sigma))(d\theta^*_{\alpha}(\sigma))\exp\left(\frac{i}{2}\int_0^T d\sigma\left(\theta^*_{\alpha}\frac{d}{d\sigma}\theta_{\alpha}+\theta_{\alpha}\frac{d}{d\sigma}\theta^*_{\alpha})(\sigma)\right)\right.\\ &\quad\times\frac{1}{D}\left(\sum_{\alpha=1}^{D}(\theta_{\alpha}(\sigma)\theta^*_{\alpha}(T))\right)\times\exp\left[i\int_0^T d\sigma\left(\theta^*_{\alpha}(\sigma)\sigma^{\alpha\beta}\theta_{\beta}(\sigma)\right)\Gamma^{\mu}_{\alpha\beta}(C(\sigma))\dot{C}_{\mu}(\sigma)\right]\\ &=\frac{1}{D}\mathrm{Tr}_{SO(D)}\left\{\exp\left(iM_{\mathrm{eff}}\left[\int_0^T d\sigma\Gamma^{\mu}(C(\sigma)\dot{C}_{\mu}(\sigma)\right]\right)_{\alpha\beta}\right\}\\ &=\frac{1}{D}\mathrm{Tr}_{SO(D)}\left\{\exp i\frac{M_{\mathrm{eff}}}{C}\left[\int_0^T d\sigma\frac{\partial^2\varphi^{(1)}_{\mu}}{\partial x^{\alpha}\partial x^{\beta}}(C(\sigma))\dot{C}_{\mu}(\sigma)\right]\right\}.\end{aligned} \tag{10}$$

Here we have used the gravitational charge (mass) of our static pairs circulating around the loop $C_{(R,T)}$ through a cumulant (leading order) evaluation of the Grasmanian variables. Namely

$$M_{\mathrm{eff}} = \int_{\substack{\theta_{\alpha}(0)=\theta_{\alpha}(T)\\ \theta^*_{\alpha}(0)=\theta^*_{\alpha}(T)}} D^F[\theta_{\alpha}(\sigma)]D^F[\theta^*_{\alpha}(\sigma)]\left(\sum_{\alpha=1}^{D}\left(\theta_{\alpha}(\sigma)\theta^*_{\alpha}(T)\right)\right)\times\left(\theta^*_{\alpha}(\sigma)\sigma^{\alpha\beta}\theta_{\beta}(\sigma)\right). \tag{11}$$

As a consequence, one should expect that (at least for large dimensionality $D \to \infty$), the effective Holonomy Factor can be written as follow in the Fourier Space

$$W[C_{(R,T)}] = \exp\left\{ i M_{\text{eff}} \left[\int d^D k \widetilde{\varphi}_\alpha(-k) k^2 j_\alpha(k, C_{(R,T)}) \right] \right\} + O\left(\frac{1}{D}\right). \tag{12}$$

Here the Fourier Transformed scalar immersion Nash field $\varphi_\mu^{(1)}$ is explicitly given by

$$\widetilde{\varphi}_\alpha(-k) = \frac{1}{(2\pi)^{D/2}} \int d^D k e^{i k_\beta x_\beta} \varphi_\alpha^{(1)}(x). \tag{13}$$

We have used the dimensional regularization rule of Bollini-Giambiagi for handling the $SO(D)$ indexes inside the ordinary integrals $k_\alpha k_\beta = \frac{k^2}{D} \delta_{\alpha\beta}$ and the contour form factor inside (11) is given explicitly by

$$j_\alpha(k, C_{(R,T)}) = \frac{1}{D} \left[\oint_{C_{(R,T)}} e^{-i k_\mu C_\mu(\sigma)} \dot{C}_\alpha(\sigma) d\sigma \right]. \tag{14}$$

After inserting all the above results into our effective sixth-order Gaussian path-integral (8), one obtains the following expression for the Newton potential in our Bosonized-metric framework of Nash immersions for Quantum Phenomenological Gravity

$$V(R) = \lim_{T \to \infty} \left\{ -\frac{M_{\text{eff}^2}}{T} \left[\int \frac{d^D k}{(2\pi)^D} \frac{|j_\alpha(k, C_{(R,T)}|^2}{k^2} \right] \right\}. \tag{15}$$

This potential can be explicitly evaluated and leading to the Newton Law of Gravitation in this phenomenological scheme for quantizing Riemann metric fields

$$V(R) = -\left(4\pi |M_{\text{eff}}|^2 G_N \cdot \frac{1}{R} \right). \tag{16}$$

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